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Exercises for 'Functional Analysis 2' [MATH-404]

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Ex 10.1 (Partial derivatives and C^1-functions)
Let $(X_j)_{j=1}^n, Y$ be Banach spaces and $X = \prod_{j=1}^n X_j$ be equipped with the norm $\max_j \|x_j\|_{X_j}$. Let $U \subset \prod_{j=1}^n X_j$ be open and $F : U \to Y$. Show that if all partial derivatives exist and $\partial_{x_j} F \in C(U, \mathcal{L}(X_j, Y))$, then $F \in C^1(U, Y)$.
Hint: Guess the form of the derivative using a result in the lecture notes.

Solution 10.1 : We first show that F is differentiable in any point $(x_1, \dots, x_n) \in U$ with

$$F'(x_1, \dots, x_n)(h_1, \dots, h_n) = \sum_{j=1}^n \partial_{x_j} F(x_1, \dots, x_n) h_j \quad \forall (h_1, \dots, h_n) \in X.$$

First note that the right-hand side indeed defines a linear, bounded operator in $\mathcal{L}(X, Y)$. Before we continue, let us introduce some notation to save space. For $j \in \{1, \dots, n\}$ set $H_j = (h_1, \dots, h_j, 0, \dots, 0) \in X$ and $x = (x_1, \dots, x_n) \in X$. By the triangle inequality we have that

$$\|F(x + H_n) - F(x) - \sum_{j=1}^n \partial_{x_j} F(x) h_j\| \le \sum_{j=1}^n \|F(x + H_j) - F(x + H_{j-1}) - \partial_{x_j} F(x) h_j\|$$

Now for each j define $g_j : [0,1] \to Y$ by $g_j(t) = F(x + t(H_j - H_{j-1}) + H_{j-1}) - t\partial_{x_j} F(x) h_j$. Note that function g varies F only in the jth component. Hence the existence of the partial derivatives implies that g is differentiable and by the chain rule (recall that $H_j - H_{j-1} = h_j$) we have that

$$g_j'(t) = \partial_{x_j} F(x + t(H_j - H_{j-1}) + H_{j-1}) h_j - \partial_{x_j} F(x) h_j.$$

Hence from the mean value theorem we deduce that

$$\begin{aligned}
\|F(x + H_j) - F(x + H_{j-1}) - \partial_{x_j} F(x) h_j\| &= \|g_j(1) - g_j(0)\| \le \sup_{t \in [0,1]} \|g_j'(t)\| \\
&= \sup_{t \in [0,1]} \|\partial_{x_j} F(x + t(H_j - H_{j-1}) + H_{j-1}) - \partial_{x_j} F(x)\|_{\mathcal{L}(X_j, Y)} \|h_j\| \\
&\le \sup_{\|H\| \le \|H_n\|} \|\partial_{x_j} F(x + H) - \partial_{x_j} F(x)\|_{\mathcal{L}(X_j, Y)} \|H_n\|,
\end{aligned}$$

where we used that $\|H_{j-1} + t(H_j - H_{j-1})\| \le t\|H_j\| + (1-t)\|H_{j-1}\| \le \|H_n\|$. This implies that

$$\frac{\|F(x + H_n) - F(x) - \sum_{j=1}^n \partial_{x_j} F(x) h_j\|}{\|H_n\|} \le \sum_{j=1}^n \sup_{\|H\| \le \|H_n\|} \|\partial_{x_j} F(x + H) - \partial_{x_j} F(x)\|_{\mathcal{L}(X_j, Y)};$$

by the continuity of the partial derivatives, we deduce differentiability of F by taking the limit $\|H_n\| \to 0$ in the above inequality.
Finally, in order to show the continuity of the derivative, take any $H \in X$ with $\|H\| \leq 1$. Then by definition of the norm on X we have $\|h_j\| \leq 1$ for all j, where $H = (h_1, \ldots, h_n)$. Hence

$$\begin{aligned}\|F'(x)H - F'(x_0)H\| &\leq \sum_{j=1}^{n} \|\partial_{x_j} F(x)h_j - \partial_{x_j} F(x_0)h_j\| \\ &\leq \sum_{j=1}^{n} \|\partial_{x_j} F(x) - \partial_{x_j} F(x_0)\|_{\mathcal{L}(X_j,Y)} \|h_j\| \leq \sum_{j=1}^{n} \|\partial_{x_j} F(x) - \partial_{x_j} F(x_0)\|_{\mathcal{L}(X_j,Y)}.\end{aligned}$$

Passing to the sumpremum over H in the left-hand side we infer that

$$\|F'(x) - F'(x_0)\|_{\mathcal{L}(X,Y)} \leq \sum_{j=1}^{n} \|\partial_{x_j} F(x) - \partial_{x_j} F(x_0)\|_{\mathcal{L}(X_j,Y)}.$$

The right-hand side tends to 0 when $x \to x_0$ by continuity of the partial derivatives. Hence F' is continuous and therefore $F \in C^1(U, Y)$.

Ex 10.2 (Consequences of Banach's fixed point theorem*)
Let X be a Banach space. Prove the following two statements :

a) Let $T : X \to X$. If there exists $\theta \in (0, 1)$ such that $\|T(x) - T(y)\| \leq \theta\|x - y\|$ for all $x, y \in X$, then $I - T$ is a homeomorphism from X to X.

b) Let $S : \overline{B_\delta(0)} \subset X \to X$ and assume that there exists $\theta \in (0, 1)$ such that

$$\|S(x) - S(y)\| \leq \theta\|x - y\| \text{ for all } x, y \in \overline{B_\delta(0)}.$$

If $\|S(0)\| < \delta(1 - \theta)$, then $I + S$ has a unique zero. Moreover,

$$B_\rho(0) \subset (I + S)(B_\delta(0))$$

for $\rho = (1 - \theta)\delta - \|S(0)\|$.

Ex 10.3 (Square root of an operator)
Let E be a Banach space and put $X = \mathcal{L}(E, E)$. Consider the function $F\colon X \to X$ such that $F(T) = T \circ T$ (which we can informally write as $F(T) = T^2$). Show that there exists a neighborhood U of I_E (the identity operator on E) and a differentiable map $G\colon U \to X$ such that $G(T)^2 = T$ for all $T \in U$.

Solution 10.3 : First, we check that F is differentiable on X. For this, let $T, S \in X$ and $\varepsilon \in \mathbb{R}$. Using linearity of T and S, and the properties of composition operation '$\circ$', for any $v \in E$ we arrive at

$$\begin{aligned}F(T + \varepsilon S)v - F(T)v &= (T + \varepsilon S) \circ (T + \varepsilon S)v - (T \circ T)v \\ &= \varepsilon(T \circ S)v + \varepsilon(S \circ T)v + \varepsilon^2(S \circ S)v\end{aligned}$$

thus the derivative of F has to read

$$F'(T)S = T \circ S + S \circ T.$$

It is easy to verify that $F'(T) \in \mathcal{L}(X)$; using the fact that $\|A \circ B\| \leq \|A\|\|B\|$ whenever A and B are linear operators (by the definition of the operator norm), we have

$$\|F(T + S) - F(T) - F'(T)S\|_X = \|(S \circ S)\|_X \leq \|S\|_X^2,$$

so that $F'(T)$ is indeed the Fréchet derivative of F at T. Note also that the mapping $T \mapsto F'(T)$ belongs to $\mathcal{L}(X, \mathcal{L}(X))$, so it is continuous.

We have $F'(I_E)S = 2S$ for all $S \in X$, which has the inverse given by $S \mapsto (1/2)S$. Therefore, from the inverse function theorem there exists an open neighborhood U of I_E and a C^1 function $G\colon U \to X$ such that

$$(F \circ G)(T) = T$$

for all $T \in U$. By the definition of F, the above identity reads as

$$G(T) \circ G(T) = T \quad \text{for all } T \in U.$$

Ex 10.4 (Small-norm solutions of nonlinear BVP)
Consider the nonlinear boundary value problem (BVP for short)

$$u'' + \lambda e^u = 0 \quad \text{in } (0, \pi), \quad u(0) = 0 = u(\pi).$$

Applying the implicit function theorem to the map $F\colon X \times \mathbb{R} \to Z$, $F(u, \lambda) = u'' + \lambda \exp(u)$, where

$$X := \{u \in C^2([0, \pi]) : \ u(0) = 0 = u(\pi)\} \quad \text{with norm } \|u\|_X = \sup_{t \in [0,\pi]} |u''(t)|,$$
$$Z := C([0, \pi]),$$

prove that for λ in a neighborhood of 0 this problem has a unique small-norm solution that depends continuously on λ.
Hint: You can use without proof that $(X, \| \cdot \|_X)$ is a Banach space.
Comment : in the above, the "small-norm" condition is specified, since otherwise uniqueness is not clear. Using convexity arguments, one could show that the solution is globally unique provided that $\lambda \leq 0$.

Solution 10.4 : It is straightforward to check that $F(u, 0) = 0$ iff $u = 0$ on $[0, \pi]$. Therefore, for $\lambda = 0$, the zero function is the unique solution to the BVP ; we want to use implicit function theorem (for the map $F : X \times \mathbb{R} \to Z$, around the point $(0, 0)$), to deduce the existence of paramaters $r > 0$, $\delta > 0$ such that there exists a unique map $T : (-r, r) \to \overline{B_\delta(0)} \subset X$ such that $F(\lambda, T(\lambda)) = 0$ for all $\lambda \in (-r, r)$.
In the setting of our BVP problem, this means that, for any λ with $|\lambda| < r$, there exists a solution $u^\lambda := T(\lambda)$ which solves the BVP problem, which moreover is unique (by the uniqueness of T) in the class of solutions u such that $\|u\|_X \leq \delta$.
Comment : this is why we talk about uniqueness for "small-norm" solutions, this condition naturally arises from an application of the implicit function theorem ; it doesn't prevent the existence of other solutions $\tilde{u}$, as long as they satisfy $\|\tilde{u}\|_X > \delta$.

We must verify the assumptions of the implicit function theorem.
To prove that F is continuous, note that

$$\|F(u, \lambda) - F(v, \mu)\|_Z \leq \|u - v\|_X + |\lambda - \mu| \cdot \|e^u\|_Z + |\mu| \cdot \|e^u - e^v\|_Z. \tag{1}$$

To show that the right-hand side goes to zero as $(u, \lambda) \to (v, \mu)$ in $X \times \mathbb{R}$, we will first show for any $u \in X$ we can control the norm $\|u\|_Z$ with $\|u\|_X$. Indeed, integrating u'' twice from 0 to π and using the boundary condition $u(0) = 0$, we find

$$u(t) = tu'(0) + \int_0^t \int_0^\tau u''(s) \, \mathrm{d}s \, \mathrm{d}\tau.$$

Now, the boundary condition $u(\pi) = 0$ yields

$$u(t) = -\frac{t}{\pi}\int_0^\pi \int_0^\tau u''(s)\,\mathrm{d}s\,\mathrm{d}\tau + \int_0^t \int_0^\tau u''(s)\,\mathrm{d}s\,\mathrm{d}\tau \tag{$\star$}$$

and thus there is a constant $C > 0$ such that

$$\|u\|_Z \le C \sup_{t\in[0,\pi]} |u''(t)| = C\|u\|_X. \tag{2}$$

Having established this fact, given a sequence $(u_n, \lambda_n) \to (u, \lambda)$ in $X \times \mathbb{R}$, then necessarily $\|u_n - u\|_Z \to 0$. As a consequence, there exists a constant R such that $|u(x)| + |u_n(s)| \le R$ for all $s \in [0, \pi]$ and $n \in \mathbb{N}$; since $x \mapsto e^x$ is a continuous function, it is uniformly continuous and bounded for $|x| \le R$; denote by ω its modulus of continuity on this set. It then follows from (1) that

$$\|F(u_n, \lambda_n) - F(u, \lambda)\|_Z \le \|u - u_n\|_X + |\lambda - \mu|\, e^R + \sup_n |\lambda_n|\, \omega(\|u_n - u\|_Z).$$

Passing to the limit as $n \to \infty$, we deduce continuity of F.

Comment : for didactical purposes, we proved continuity of F in a self-contained matter. But if one directly proves Fréchet differentiability of F (we will shortly do it only in the u-variable), then one can automatically deduce continuity of F from the result of the notes

Furthermore, for $u, v \in X$, $\lambda \in \mathbb{R}$, and $\varepsilon \in \mathbb{R} \setminus \{0\}$

$$\frac{1}{\varepsilon}\big[F(u + \varepsilon v, \lambda) - F(u, \lambda)\big] = v'' + \lambda e^u \frac{e^{\varepsilon v} - 1}{\varepsilon};$$

taking the limit $\varepsilon \to 0$ we expect that

$$D_u F(u, \lambda)v = v'' + \lambda e^u v. \tag{3}$$

Informed with the educated guess (3) for $D_u F$, we now verify that this is indeed the case. It is convenient to notice that $F(u, \lambda) = F_1(u) + \lambda F_2(u)$, for $F_1(u) = u''$ and $F_2(u) = e^u$; since F_1 is linear, it is also Fréchet with $D_u F_1(\lambda, u)(v) = F_1(v)$, so we only need to check differentiability of F_2. By the same argument as above, we expect $D_u F_2(\lambda, u) = e^u v$. For any u, $v \in X$, it holds

$$|e^{u(t)+v(t)} - e^{u(t)} - e^{u(t)}v(t)| \le |v(t)| \sup_{h\in[0,v(t)|} |e^{u(t)+h} - e^{u(t)}|$$

where we used the mean valued theorem and the fact that $(e^x)' = e^x$. It follows that, for $\|v\|_Z \le \varepsilon$, it holds

$$\|e^{u+v} - e^u - e^u v\|_Z \le \|v\|_Z \sup_{h\in[0,\|v\|_Z|} \omega(|h|)$$

where we also used again the uniform continuity of e^x (with modulus ω) on a suitable bounded set. We deduce that

$$\frac{\|e^{u+v} - e^u - e^u v\|_Z}{\|v\|_Z} \le \omega(\|v\|_Z)$$

and so (using (2)) we conclude that

$$\lim_{\|v\|_X \to 0} \frac{\|e^{u_0+v} - e^{u_0} - e^{u_0} v\|_Z}{\|v\|_X} = 0,$$

which verifies Fréchet differentiability of F_2 and overall the existence of $D_u F$ with formula (3). In view of (2) and (3), it's easy to check that $D_u F(u, \lambda)$ is a bounded linear operator, with

$$\|D_u F(u, \lambda)\| \leq 1 + |\lambda| e^{C\|u\|_X}.$$

Finally, it remains to verify continuity of $(u, \lambda) \mapsto D_u F(u, \lambda)$. Arguing similarly to above, for any (u_1, λ_1), (u_2, λ_2) it holds

$$\|D_u F(u_1, \lambda_1) - D_u F(u_2, \lambda_2)\| \leq \|\lambda_1 e^{u_1} - \lambda_2 e^{u_2}\|_Z \leq |\lambda_1 - \lambda_2| e^{\|u_1\|_Z} + |\lambda_2| \|e^{u_1} - e^{u_2}\|_Z;$$

from here, arguing in the same way as when we established continuity of F, we can conclude that $(u, \lambda) \mapsto D_u F(u, \lambda)$ is a continuous map.

Comment : notice that, although we performed the argument for $x \mapsto e^x$, it would have worked for any C^1-regular map $f : \mathbb{R} \to \mathbb{R}$. In particular, going through similar computations, $G(u) := f \circ u$ as a map from Z to Z is a C^1 map with Fréchet differential $DG(u)(v) = f'(u)v$.

Let us now consider the linear mapping $T = D_u F(0, 0) : X \to Z$. We must show that T is invertible. To see this, note that for any $h \in Z$

$$Tv = h \quad \iff \quad v'' = h \quad \text{in } (0, \pi), \quad v(0) = 0 = v(\pi).$$

Substituting v for u and h for u'' in $(\star)$, we obtain

$$v(t) = (T^{-1}h)(t) = \int_0^\pi G(t, s) h(s) \, \mathrm{d}s,$$

where

$$G(t, s) = \begin{cases} -\dfrac{1}{\pi}(\pi - t)s, & 0 \leq s \leq t, \\[2ex] -\dfrac{1}{\pi}t(\pi - s), & t \leq s \leq \pi. \end{cases}$$

By the definition of $\|\cdot\|_X$, it is also immediate to see that

$$\|T^{-1}h\|_X = \|v''\|_Z = \|h\|_Z$$

so that T^{-1} is continuous.

Hence, all conditions of the implicit function theorem are satisfied and we may conclude that, for all λ sufficiently small, $F(u, \lambda) = 0$ has a unique small-norm solution $u \in X$, and that the map $\lambda \mapsto u(\lambda)$ is continuous.